ON THE SUM OF THE INDEX OF A PARABOLIC SUBALGEBRA AND OF ITS NILPOTENT RADICAL

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Abstract. In this short note, we investigate the following question of Panyushev stated in [10]: "Is the sum of the index of a parabolic subalgebra of a semisimple Lie algebra $\mathfrak{g}$ and the index of its nilpotent radical always greater than or equal to the rank of $\mathfrak{g}$?". Using the formula for the index of parabolic subalgebras conjectured in [13] and proved in [3, 6], we give a positive answer to this question. Moreover, we also obtain a necessary and sufficient condition for this sum to be equal to the rank of $\mathfrak{g}$. This provides new examples of direct sum decomposition of a semisimple Lie algebra verifying the "index additivity condition" as stated by Raïs in [11].

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero. For $f \in \mathfrak{g}^*$, we denote by $\mathfrak{g}^f = \{X \in \mathfrak{g}; f([X,Y]) = 0$ for all $Y \in \mathfrak{g}\}$, the annihilator of f for the coadjoint representation of $\mathfrak{g}$. The *index* of $\mathfrak{g}$, denoted by $\chi(\mathfrak{g})$, is defined to be

$$\chi(\mathfrak{g}) = \min_{f \in \mathfrak{g}^*} \dim \mathfrak{g}^f.$$

It is well-known that if $\mathfrak{g}$ is an algebraic Lie algebra and G denote its algebraic adjoint group, then $\chi(\mathfrak{g})$ is the transcendence degree of the field of G-invariant rational functions on $\mathfrak{g}^*$.

The index of a semisimple Lie algebra $\mathfrak{g}$ is equal to the rank of $\mathfrak{g}$. This can be obtained easily from the isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form. There has been quite a lot of recent work on the determination of the index of certain subalgebras of a semisimple Lie algebra: parabolic subalgebras and related subalgebras ([2], [9], [13], [8]), centralizers of elements and related subalgebras ([10], [1], [15], [7]).

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{u}$ (resp. $\mathfrak{l}$) the nilpotent radical (resp. a Levi factor) of $\mathfrak{p}$. In [10,

Corollary 1.5 (i)], Panyushev showed that

$$\chi(\mathfrak{p}) + \chi(\mathfrak{u}) \leq \dim \mathfrak{l}. \tag{1}$$

He then suggested [10, Remark (ii) of Section 6] that

$$\chi(\mathfrak{p}) + \chi(\mathfrak{u}) \geq \operatorname{rk} \mathfrak{g}. \tag{2}$$

For example, it is well-known that if $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ is its nilpotent radical, then $\chi(\mathfrak{b}) + \chi(\mathfrak{n}) = \operatorname{rk} \mathfrak{g}$ (see for example [12], [14, Chapter 40]). It is therefore also interesting to characterise parabolic subalgebras where equality holds in (2). Indeed, in [11], Raïs looked for examples of direct sum decompositions $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n}$ verifying the "index additivity condition", namely $\mathfrak{m}$ and $\mathfrak{n}$ are Lie subalgebras of $\mathfrak{g}$ and

$$\chi(\mathfrak{g}) = \chi(\mathfrak{m}) + \chi(\mathfrak{n}).$$

If $\mathfrak{u}_-$ denotes the nilpotent radical of the opposite parabolic subalgebra $\mathfrak{p}_-$ of $\mathfrak{p}$, then $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}_-$ and the Lie algebras $\mathfrak{u}$ and $\mathfrak{u}_-$ are isomorphic. Thus parabolic subalgebras such that equality holds in (2) would provide examples of direct sum decompositions verifying the index additivity condition.

Using the formula, conjectured in [13] and proved in [3, 6], for the index of parabolic subalgebras, we obtain a formula for the sum $\chi(\mathfrak{p}) + \chi(\mathfrak{u})$. By a careful analysis of root systems, we prove the inequality (2) and give a necessary and sufficient condition of equality to hold in (2) (See Theorem 2.2).

To describe the index of a parabolic subalgebra, and the index of its nilpotent radical we need to recall Kostant's cascade construction of pairwise strongly orthogonal roots ([4], [5], [14]).

Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$. Denote by R, R^+ and $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ respectively the set of roots, positive roots and simple roots with respect to $\mathfrak{h}$ and $\mathfrak{b}$. For any $\alpha \in R$, let $\mathfrak{g}_\alpha$ be the root subspace associated to α. Choose X_α such that $\alpha([X_\alpha, X_{-\alpha}]) = 2$. We shall write $\alpha^\vee = [X_\alpha, X_{-\alpha}] \in \mathfrak{h}$, and for $\lambda \in \mathfrak{h}^*$, $\langle \lambda, \alpha^\vee \rangle = \lambda(\alpha^\vee)$. For $S \subset \Pi$, we denote by $R_S = R \cap \mathbb{Z}S$, $R_S^+ = R_S \cap R^+$. If S is connected, then we shall denote by ε_S the highest root of R_S.

Let $S \subset \Pi$. We define $\mathcal{K}(S)$ inductively as follows:

a) $\mathcal{K}(\emptyset) = \emptyset$.
b) If $S_1, \dots, S_r$ are the connected components of S, then $\mathcal{K}(S) = \mathcal{K}(S_1) \cup \cdots \cup \mathcal{K}(S_r)$.
c) If S is connected, then $\mathcal{K}(S) = \{S\} \cup \mathcal{K}(\widehat{S})$ where $\widehat{S} = \{\alpha \in S; \langle \alpha, \varepsilon_S^\vee \rangle = 0\}$.

It is well-known that (see for example [14, Chapter 40]) elements of $\mathcal{K}(S)$ are connected subsets of S. Moreover, if we denote by $\mathcal{R}(S) = \{\varepsilon_K; K \in \mathcal{K}(S)\}$, then $\mathcal{R}(S)$ is a maximal set of pairwise strongly orthogonal roots in R_S.

Let us also recall the following properties of $\mathcal{K}(S)$:

Lemma 1.1. *Let S be a subset of Π, $K, K' \in \mathcal{K}(S)$ and set*

$$\begin{aligned}\Gamma^K &= \{\alpha \in R_K; \langle \alpha, \varepsilon_K^\vee \rangle > 0\} \\ &= \{\alpha = \textstyle\sum_{\beta \in K} n_\beta \beta \in R_K^+; n_\beta > 0 \text{ for some } \beta \in K \setminus \widehat{K}\}.\end{aligned}$$

i) *We have either $K \subset K'$ or $K' \subset K$ or K and K' are connected components of $K \cup K'$.*

ii) *$\Gamma^K = R_K^+ \setminus \{\beta \in R_K^+; \langle \beta, \varepsilon_K^\vee \rangle = 0\}$. In particular, R_K^+ is the disjoint union of the Γ^K's, $K \in \mathcal{K}(S)$.*

iii) *$\sum_{\alpha \in \Gamma^K} \mathfrak{g}_\alpha$ is a Heisenberg Lie algebra whose centre is $\mathfrak{g}_{\varepsilon_K}$. Thus if $\alpha, \beta \in \Gamma^K$ verify $\alpha + \beta \in R$, then $\alpha + \beta = \varepsilon_K$.*

iv) *Suppose that $\alpha \in \Gamma^K$ and $\beta \in \Gamma^{K'}$ verify $\alpha + \beta \in R$, then either $K \subset K'$ and $\alpha + \beta \in \Gamma^{K'}$ or $K' \subset K$ and $\alpha + \beta \in \Gamma^K$.*

Table 1.2. The cardinality of $\mathcal{K}(\Pi)$ is listed below for an irreducible root system R:

Type	$A_\ell, \ell \geq 1$	$B_\ell, C_\ell, \ell \geq 2$	$D_\ell, \ell \geq 4$	E_6	E_7	E_8	F_4	G_2
$\sharp\mathcal{K}(\Pi)$	$\left[\dfrac{\ell+1}{2}\right]$	ℓ	$2\left[\dfrac{\ell}{2}\right]$	4	7	8	4	2

where for any $x \in \mathbb{Q}$, $[x]$ is the unique integer such that $[x] \leq x < [x]+1$.

Examples 1.3. Let R be an irreducible root system. We shall use the numbering of simple roots in [14, Chapter 18]. Set $k = \sharp\mathcal{K}(\Pi)$.

1) Let R be of type A_ℓ. Then $\mathcal{K}(\Pi) = \bigcup_{i=1}^k \{K_i\}$ where $K_i = \{\alpha_i, \dots, \alpha_{\ell+1-i}\}$. For $1 \leq i \leq k$,

$$\Gamma^{K_i} = \{\alpha_i + \cdots + \alpha_{i+r}\ ,\ \alpha_{\ell+1-i-r} + \cdots + \alpha_{\ell+1-i}; 0 \leq r \leq \ell - 2i\} \cup \{\varepsilon_{K_i}\}.$$

2) Let R be of type D_{2n+1}. Then $k = 2n$,

$$\mathcal{K}(\Pi) = \bigcup_{i=1}^n \{K_i\} \cup \{L_i\}$$

where $K_i = \{\alpha_{2i-1}, \dots, \alpha_{2n+1}\}$ and $L_i = \{\alpha_{2i-1}\}$. For $1 \leq i \leq n$,

$$\Gamma^{K_i} = \left\{ \sum_{j=2i-1}^{\ell} m_j \alpha_j; m_{2i} \neq 0 \right\}\ ,\ \Gamma^{L_i} = \{\alpha_{2i-1}\}.$$

3) Let R be of type E_6. Then

$$\mathcal{K}(\Pi) = \{\Pi\} \cup \{\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}\} \cup \{\{\alpha_3, \alpha_4, \alpha_5\}\} \cup \{\{\alpha_4\}\}.$$

2. Main result

Recall that for any subset $S \in \Pi$,

$$\mathfrak{p}_S = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_S \cup R^+} \mathfrak{g}_\alpha$$

is a (standard) parabolic subalgebra of $\mathfrak{g}$. Any parabolic subalgebra of $\mathfrak{g}$ is conjugated to a standard parabolic subalgebra. The Lie subalgebra

$$\mathfrak{u}_S = \bigoplus_{\alpha \in R^+ \setminus R_S} \mathfrak{g}_\alpha$$

is the nilpotent radical of $\mathfrak{p}_S$.

For $S \subset \Pi$, denote by V_S the vector subspace of $\mathfrak{h}^*$ spanned by the elements of $\mathcal{R}(S)$ and $\mathcal{R}(\Pi)$. Set

$$\mathcal{E}_S = \{K \in \mathcal{K}(\Pi); X_{\varepsilon_K} \in \mathfrak{u}_S\} = \{K \in \mathcal{K}(\Pi); \varepsilon_K \notin R_S\}$$
$$\text{and} \quad \mathcal{Q}_S = \left(\bigcup_{K \in \mathcal{E}_S} \Gamma^K \right) \cap R_S^+.$$

The subset $\mathcal{E}_S$ has the following simple characterisation.

Lemma 2.1. *Let T_S be the union of $K \in \mathcal{K}(\Pi)$ verifying $K \subset S$. Then $\mathcal{E}_S = \mathcal{K}(\Pi) \setminus \mathcal{K}(T_S)$.*

Proof. This is straightforward. □

Our main result is the following theorem.

Theorem 2.2. *Let $S \subset \Pi$. Then*

$$\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g} + \sharp\mathcal{K}(S) - \sharp\mathcal{K}(T_S) + 2(\sharp\mathcal{K}(\Pi) - \dim V_S) + \sharp\mathcal{Q}_S. \tag{3}$$

We have $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) \geq \operatorname{rk}\mathfrak{g}$, and equality holds if and only if the following conditions are satisfied:

i) $\sharp(\mathcal{K}(S) \cup \mathcal{K}(\Pi)) = \dim V_S$.

ii) *For any connected component S' of S, we have either $S' \in \mathcal{K}(\Pi)$ or $\sharp(S' \setminus T_S) = 1$.*

Proof. The formula for the sum $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S)$ is a direct consequence of the formula of the index of parabolic subalgebras conjectured in [13] and proved in [3, 6] :

$$\chi(\mathfrak{p}_S) = \operatorname{rk}\mathfrak{g} + \sharp\mathcal{K}(\Pi) + \sharp\mathcal{K}(S) - 2\dim V_S \ , \tag{4}$$

and the formula for the index of $\mathfrak{u}_S$ (see for example [14, Chapter 40]), which, in view of Lemma 1.1, can be expressed in the following way:

$$\chi(\mathfrak{u}_S)=\sharp\mathcal{E}_S+\sum_{K\in\mathcal{E}_S}\sharp\Gamma^K-\dim\mathfrak{u}_S=\sharp\mathcal{E}_S+\sharp\mathcal{Q}_S \tag{5}$$

To prove the rest of the theorem, we may clearly assume that $\mathfrak{g}$ is simple. Observe that

$$\sharp\mathcal{K}(S)-\sharp\mathcal{K}(T_S)\geq 0. \tag{6}$$

1) Let $S_1,\dots,S_r$ be the connected components of S. For each i, there is a unique $K_i\in\mathcal{K}(\Pi)$ (see Lemma 1.1) such that $\varepsilon_{S_i}\in\Gamma^{K_i}$. If $K_i=S_i$, then S_i is a connected component of T_S. Otherwise $K_i\in\mathcal{E}_S$, and we have

$$\varepsilon_{S_i}\in\Gamma^{K_i}\cap\Gamma^{S_i}\subset\mathcal{Q}_S.$$

2) It follows from Point 1) that $\mathcal{Q}_S=\emptyset$ if and only if $\mathcal{K}(S)\subset\mathcal{K}(\Pi)$ (or equivalently $S=T_S$).

3) Note that the connected components of T_S are the connected components of $T_S\cap S_i$, it follows again from Point 1) that:

$$\begin{aligned}\sharp\mathcal{K}(S)-\sharp\mathcal{K}(T_S) &= \sum_{i=1}^{r}(\sharp\mathcal{K}(S_i)-\sharp\mathcal{K}(T_S\cap S_i))\\ &= \sum_{K_i\in\mathcal{E}_S}(\sharp\mathcal{K}(S_i)-\sharp\mathcal{K}(T_S\cap S_i)).\end{aligned}$$

4) From Table 1.2, we have $\dim V_S=\sharp\mathcal{K}(\Pi)=\operatorname{rk}\mathfrak{g}$ in the cases where $\mathfrak{g}$ is of type B_ℓ, C_ℓ, D_{2n}, E_7, E_8, F_4 et G_2. The inequality follows immediately from (3) and (6), and the condition for equality follows from Point 2).

5) Type A_ℓ.

For any i verifying $S_i\neq K_i$, by Point 1), Lemma 1.1 and Examples 1.3, half of $\Gamma^{S_i}\setminus\{\varepsilon_{S_i}\}$ belongs to $\mathcal{Q}_S$. Since $\sharp(\Gamma^{S_i})=2\sharp(S_i)-1$ (Examples 1.3), such an S_i contributes $\sharp(S_i)$ elements of $\mathcal{Q}_S$.

Again, since we are in type A, $\mathcal{K}(\Pi)$ is totally ordered by inclusion. It follows that T_S is connected. Without loss of generality, we may assume that $T_S\subset S_1$.

Suppose that $S_1=T_S$. Then from the previous discussion, we deduce that

$$\chi(\mathfrak{p}_S)+\chi(\mathfrak{u}_S)\geq\operatorname{rk}\mathfrak{g}+\sharp\mathcal{K}(S\setminus S_1)+2(\sharp\mathcal{K}(\Pi)-\dim V_S)+\sharp(S\setminus S_1).$$

But our hypothesis implies that

$$\dim V_S\leq\sharp\mathcal{K}(\Pi)+\sharp\mathcal{K}(S\setminus S_1)=\sharp(\mathcal{K}(\Pi)\cup\mathcal{K}(S)), \tag{7}$$

so

$$\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) \geq \operatorname{rk}\mathfrak{g} + \sharp(S \setminus S_1) - \sharp\mathcal{K}(S \setminus S_1).$$

Hence $\chi(\mathfrak{p}_S)+\chi(\mathfrak{u}_S) \geq \operatorname{rk}\mathfrak{g}$. For equality to hold, we must have equality in (7) and

$$\sharp(S \setminus S_1) = \sharp\mathcal{K}(S \setminus S_1).$$

This latter is only possible if $\sharp(S_i) = 1$ for $i \geq 2$, so we have conditions (i) and (ii). Conversely, suppose that conditions (i) and (ii) are verified, then $\sharp(S_i) = 1$ for $i \geq 2$. Consequently $\mathcal{Q}_S = S \setminus S_i$ by Point 1) and the definition of $\mathcal{Q}_S$.

Suppose that $S_1 \supsetneq T_S$ (this includes the case $T_S = \emptyset$). Then $\mathcal{K}(S) \cap \mathcal{K}(\Pi) = \emptyset$. Thus

$$\dim V_S \leq \sharp\mathcal{K}(\Pi) + \sharp\mathcal{K}(S) = \sharp(\mathcal{K}(\Pi) \cup \mathcal{K}(S)). \tag{8}$$

We deduce from Point 1) and the remark in the first paragraph of Point 5) that

$$\begin{aligned}\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) &\geq \operatorname{rk}\mathfrak{g} + \sharp\mathcal{K}(S) - \sharp\mathcal{K}(T_S) + 2(\sharp\mathcal{K}(\Pi) - \dim V_S) + \sharp S \\ &\geq \operatorname{rk}\mathfrak{g} + \sharp(S) - \sharp\mathcal{K}(S) - \sharp\mathcal{K}(T_S).\end{aligned}$$

Hence

$$\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) \geq \operatorname{rk}\mathfrak{g} + \sum_{i=1}^{r} \left(\sharp(S_i) - \left[\frac{\sharp(S_i)+1}{2}\right] \right) - \left[\frac{\sharp(T_S)+1}{2}\right].$$

Since $T_S \subsetneq S_1$, we deduce from Table 1.2 that

$$\sharp(S_1) - \left[\frac{\sharp(S_1)+1}{2}\right] - \left[\frac{\sharp(T_S)+1}{2}\right] \geq 0.$$

So we have our inequality $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) \geq \operatorname{rk}\mathfrak{g}$.

Now for the equality $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g}$ to hold, we must have equality in (8) and $\sharp\mathcal{Q}_S = \sharp(S)$,

$$\sharp(S_1) - \left[\frac{\sharp(S_1)+1}{2}\right] - \left[\frac{\sharp(T_S)+1}{2}\right] = 0 \text{ and } \sharp(S_i) - \left[\frac{\sharp(S_i)+1}{2}\right] = 0$$

for $i \geq 2$. This implies that $\sharp(S_1) = \sharp(T_S) + 1$, and $\sharp(S_i) = 1$ for $i \geq 2$. So we have conditions (i) and (ii). Conversely, if conditions (i) and (ii) are verified, then $\sharp(S_i) = 1$ for $i \geq 2$, and $\sharp(S_1) = \sharp(T_S) + 1$. In view of the above, to show that $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g}$, it suffices to check that $\sharp(\mathcal{Q}_S \cap R^+_{S_1}) = \sharp(S_1)$, which is a straightforward verification.

6) Type D_{2n+1}.

In this case, $\sharp\mathcal{K}(\Pi) = \operatorname{rk}\mathfrak{g} - 1$. Let us use the numbering of simple roots in [14, Chapter 18]. We check easily that $\alpha_1, \ldots, \alpha_{\ell-2} \in V_\Pi$.

If $\dim V_S = \sharp\mathcal{K}(\Pi)$, then the inequality follows from (3) and (6), and the condition for equality follows from Point 2).

Suppose now that $\dim V_S = \operatorname{rk}\mathfrak{g}$ and $\alpha_{\ell-1} \in S$ (the case $\alpha_\ell \in S$ being analogue). Then

$$\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g} + \sharp\mathcal{K}(S) - \sharp\mathcal{K}(T) - 2 + \sharp(\mathcal{Q}_S), \tag{9}$$

and the connected component S_1 of S containing $\alpha_{\ell-1}$ is not in $\mathcal{K}(\Pi)$ (for otherwise, we would have $\dim V_S = \sharp\mathcal{K}(\Pi)$).

By Point 1), $\varepsilon_{S_1} \in \mathcal{Q}_S$. By examining the possibilities for S_1 and K_1 (Examples 1.3), we verify that

$$\sharp\mathcal{K}(S_1) - \sharp\mathcal{K}(T_S \cap S_1) + \sharp(\Gamma^{K_1} \cap \Gamma^{S_1}) \geq 2 \tag{10}$$

with equality if and only if S_1 is of type A_1 or A_2. We have therefore obtained the inequality.

In fact, we showed in the previous paragraph that already we have

$$\operatorname{rk}\mathfrak{g} + \sharp\mathcal{K}(S_1) - \sharp\mathcal{K}(T \cap S_1) - 2 + \sharp(\Gamma^{K_1} \cap \Gamma^{S_1}) \geq \operatorname{rk}\mathfrak{g}.$$

So if $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g}$, then from (9) and the above inequality, we must have $\mathcal{K}(S \setminus S_1) \subset \mathcal{K}(\Pi)$, and also equality in (10). Hence conditions (i) and (ii). Conversely, suppose that conditions (i) and (ii) are verified, then the fact that $\alpha_1, \dots, \alpha_{\ell-2} \in V_\Pi$ implies that $\mathcal{K}(S \setminus S_1) \subset \mathcal{K}(\Pi)$ and $\sharp\mathcal{K}(S_1) = 1$. Hence S_1 is of type A_1, A_2. It is then easy to check that $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g}$.

7) <u>Type E_6</u>.

Here, we have $\sharp\mathcal{K}(\Pi) = 4$ and $\alpha_2, \alpha_4 \in V_\Pi$. Let S_1 be a connected component of S such that $\dim V_{S_1} > 4$. Under these conditions, the possibilities are:

S_1	$\dim V_{S_1}$	$T_S \cap S_1$	$\sharp\mathcal{K}(S_1)$	$\sharp\mathcal{K}(T_S \cap S_1)$	K_1	$\sharp(\Gamma^{K_1} \cap \Gamma^{S_1})$
A_1	5	$\emptyset$	1	0	A_3 or A_5	1
A_2	5	$\emptyset$	1	0	A_5	2
A_2	5	A_1	1	1	A_3	2
A_3	6	A_1	2	1	A_5	3
A_3	5	A_1	2	1	E_6	3
A_4	6	A_1	2	1	E_6	4
A_4	6	A_3	2	2	A_5	4
D_4	5	A_3	4	2	E_6	4
D_5	6	A_3	4	2	E_6	9

Thus

$$\sharp\mathcal{K}(S_1) - \sharp\mathcal{K}(T_S \cap S_1) + \sharp(\Gamma^{S_1} \cap \Gamma^{K_1}) \geq 2(\dim V_S - \sharp\mathcal{K}(\Pi)).$$

A direct verification gives the result. Note that as in the case of type A_ℓ, $\mathcal{K}(\Pi)$ is totally ordered by inclusion, so T_S is connected. $\square$

Remarks 2.3. 1. Theorem 2.2 says that if $\mathcal{K}(S) \subset \mathcal{K}(\Pi)$ or equivalently, $S' \in \mathcal{K}(\Pi)$ for any connected component S' of S, then $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g}$.

2. When $\mathfrak{g}$ is of type B_ℓ, C_ℓ, D_{2n}, E_7, E_8, F_4 or G_2, we have $\sharp\mathcal{K}(\Pi) = \operatorname{rk}\mathfrak{g}$. In these cases, the condition (i) in Theorem 2.2 is equivalent to $\mathcal{K}(S) \subset \mathcal{K}(\Pi)$, and consequently, condition (ii) is automatically satisfied.

Example 2.4. Let us consider the case of minimal parabolic subalgebras. So $S = \{\alpha\}$ and (3) is an equality. It follows that

$$\chi(\mathfrak{p}) + \chi(\mathfrak{u}) = \begin{cases} \operatorname{rk}\mathfrak{g} & \text{if } \{\alpha\} \in \mathcal{K}(\Pi), \\ \operatorname{rk}\mathfrak{g} & \text{if } \{\alpha\} \notin \mathcal{K}(\Pi) \text{ and } \dim V_S = \sharp\mathcal{K}(\Pi) + 1, \\ \operatorname{rk}\mathfrak{g} + 2 & \text{if } \{\alpha\} \notin \mathcal{K}(\Pi) \text{ and } \dim V_S = \sharp\mathcal{K}(\Pi). \end{cases}$$

Thus the minimal parabolic subalgebras $\mathfrak{p}_S$ verifying $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g}$ are (in the simple roots numbering of [14, Chapter 18]):

Type	$\mathcal{K}(S) \not\subset \mathcal{K}(\Pi)$	$\mathcal{K}(S) \subset \mathcal{K}(\Pi)$
A_ℓ	$i \neq \frac{\ell+1}{2}$	$i = \frac{\ell+1}{2}$
B_ℓ	none	i odd
C_ℓ	none	$i = \ell$
D_{2n+1}	$i = 2n, 2n+1$	$i < 2n$ odd
D_{2n}	none	i odd or $i = 2n$
E_6	$i \neq 2, 4$	$i = 4$
E_7	none	$i = 2, 3, 5, 7$
E_8	none	$i = 2, 3, 5, 7$
F_4	none	$i = 2$
G_2	none	$i = 1$

Example 2.5. In the other extremity, it is easy to check that maximal parabolic subalgebras of $\mathfrak{g} = \mathrm{sl}_{\ell+1}$ verifying $\chi(\mathfrak{p}) + \chi(\mathfrak{u}) = \operatorname{rk}\mathfrak{g}$ are exactly the ones associated to simple roots at the extremities of the Dynkin diagram.

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